Problem Set 1

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1. Padmanabhan, Problem 1.3

2. Padmanabhan, Problem 1.4

3. **Burning wood with the sun:** Consider using a converging lens – i.e., a magnifying glass – to light a piece of wood on fire. If the temperature at which wood ignites is 800 F, find the focal ratio of the lens necessary to cause ignition. The focal ratio is $f \equiv F/d$, where F is the focal length and d is the diameter of the lens. Assume that the albedo of the wood (the ratio of the total reflected flux to the total incident flux) is 50% and that the Sun's effective surface temperature is 5700 C.

4. **Modified gravitational force law:** In class, we discussed the relaxation time for gravitational interactions between stars or galaxies. Physicists today often consider altered gravitational force laws for various reasons. To get some practice working with this, let's consider a modified gravitational force law of the Yukawa form:

$$F = -\frac{Gm^2}{r^2} e^{-|r|/\lambda}, \tag{1}$$

where we have a new length scale, λ, in the problem. To make the problem simpler, assume that this scale is considerably larger than the size scale, R, of the system we're studying – say, $\lambda \sim 10R$. Then estimate how long it will take this system to relax. Is this time scale longer or shorter than in the case of normal gravity? Can you guess what the effect of an even smaller λ would be? Note: if you need to make any further approximations or assumptions to solve this problem, please do so – simply note what assumption you are making.

1.1 Why does an accelerated charge radiate? [2]

(a) The electric field of a stationary point charge falls as r^{-2}. The field of a charge, moving with a uniform velocity, can be obtained from the Coulomb field by a Lorentz transformation; show that this field also decreases as r^{-2}.

(b) But when the charge is accelerating, the electric field picks up a term which falls as r^{-1}, usually called the 'radiation field'. A field with $E \propto r^{-1}$ has an energy flux $S \propto E^2 \propto r^{-2}$. Since the surface area of a sphere increases as r^2, the same amount of energy will flow through spheres of different radii. This fact allows the accelerating charge to transfer energy to large distances and thus provides the radiation field with an independent dynamical existence.

Consider a charged particle which was at rest till $t = 0$ and was accelerated to a velocity $\mathbf{v}$ in a small interval of time Δt. Examine the electric field everywhere at some time $t \gg \Delta t$ and show that the electric field in a shell-like region of thickness $c\Delta t$, located around $r \cong ct$, has an r^{-1} dependence.

Use the above result to show that the radiation field due to a *non-relativistic* charged particle with acceleration $\mathbf{a}(t)$ can be expressed in the form

$$\mathbf{E}(t,\mathbf{r}) = \left[\frac{q}{c^2 r}(\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{a}))\right]_{\text{ret}}, \qquad \mathbf{B} = \hat{\mathbf{n}} \times \mathbf{E}, \tag{1.1}$$

where $\mathbf{r}$ is the vector from the charge to the field point and $\hat{\mathbf{n}} = (\mathbf{r}/r)$. The subscript 'ret' implies that the expression should be evaluated at the time $t' = t - (r/c)$.

(c) Consider a single charged particle moving with acceleration a. Show that the power radiated is $(d\mathcal{E}/dt) \cong (q^2 a^2/c^3)$.

Problems due on 11 Sep. 2008

1.3 Bremsstrahlung and synchrotron radiation [1]

We shall now estimate the radiation emitted in two important astrophysical contexts:

(a) In a fully ionized plasma, collisions between electrons and ions will accelerate the electrons and make them radiate. Estimate the energy radiated per second per unit volume from a hydrogen plasma at temperature T. (This process is called 'thermal Bremsstrahlung'.)

(b) Several astrophysical plasmas host magnetic fields. Charged particles, accelerated by the magnetic field, will emit radiation (called 'cyclotron' or 'synchrotron' radiation depending on whether the charge is non-relativistic or relativistic). Estimate the power radiated in this case.

1.4 Thomson scattering and Eddington limit [1]

(a) An electromagnetic wave with amplitude E hits a free charged particle and makes it oscillate with non-relativistic velocities. The oscillating charge, in turn, emits radiation in all directions. This process may be thought of as the scattering of electromagnetic radiation by a free charged particle. Estimate the scattering cross-section for this process.

(b) Matter falling towards a massive gravitating object gains kinetic energy. If this matter is brought to a halt suddenly – say, at the surface of the central body – then the kinetic energy can be converted into radiation with some efficiency ϵ. This process is invoked in several astrophysical contexts to account for large luminosities of extragalactic objects. Show that the luminosity which can be produced by such accretion is limited by the value (called the Eddington limit)

$$L_E \cong \frac{4\pi G M m_p c}{\sigma_T} \cong 10^{46}\left(\frac{M}{10^8\ M_\odot}\right)\text{erg s}^{-1}. \tag{1.3}$$

Estimate the corresponding limiting values for (i) accretion rate (dm/dt); (ii) typical time-scale for the accretion; and (iii) peak frequency of the emission of radiation. Take the size of the central object to be $R \cong (GM/c^2)$.