The vanishing of the first term defines the extremum condition and the sign of the second term decides whether the extremum is stable or merely a saddle point.

Show that among all $f(\mathbf{x}, \mathbf{v})$ with a given E, M and $\rho(\mathbf{x})$, the entropy is extremized by the Maxwellian distribution function with

$$f(\mathbf{x}, \mathbf{v}) = (2\pi T)^{-3/2}\, \rho(\mathbf{x}) \exp\left(-v^2/2T\right), \tag{2.42}$$

where $T = \left(2K/3m\right)$ and K is the kinetic energy of the system. Consider now a small variation in the spatial density $\delta\rho$ leading to a corresponding variation $\delta\phi$ in the gravitational potential. Show that S is maximized when ρ and ϕ satisfy equations (2.33). Also show that the second variation of S has the form

$$\delta^2 S = -\int d^3\mathbf{x} \left(\frac{\delta\rho\,\delta\phi}{2T} + \frac{(\delta\rho)^2}{2\rho}\right) - \frac{1}{3MT^2}\left(\int d^3\mathbf{x}\, \phi\, \delta\rho\right)^2 . \tag{2.43}$$

Whether the extremum solution to (2.33) is stable or unstable is determined by this second variation.

2.10 Isothermal sphere [2,N]

A self-gravitating system is described by equations (2.33) in the mean field limit. The two equations may be combined to obtain

$$\nabla^2\phi = 4\pi G \rho_c \exp\left(-\beta\left[\phi(\mathbf{x}) - \phi(0)\right]\right), \tag{2.44}$$

which describes an 'isothermal sphere'. Introduce length, mass and energy scales by the definitions

$$L_0 \equiv (4\pi G \rho_c \beta)^{1/2}, \quad M_0 = 4\pi\rho_c L_0^3, \quad \phi_0 \equiv \beta^{-1} = \frac{GM_0}{L_0} \tag{2.45}$$

and express the radial distance r, density ρ, mass $M(r)$ contained within a radius r and potential ϕ in terms of these variables in dimensionless form:

$$x \equiv \frac{r}{L_0}, \quad n \equiv \frac{\rho}{\rho_c}, \quad m = \frac{M(r)}{M_0}, \quad y \equiv \beta\left[\phi - \phi(0)\right]. \tag{2.46}$$

(a) Show that $n = \left(2/x^2\right), m = 2x, y = 2\ln x$ is a solution to (2.44) and that all other solutions to this equation tend to $y = 2\ln x$ for large x.

(b) Introduce the variables $v = \left(m/x\right)$ and $u = \left(nx^3/m\right)$ and show that the isothermal equations can be expressed as

$$\frac{u}{v}\frac{dv}{du} = -\frac{u-1}{u+v-3}, \tag{2.47}$$

with the boundary conditions $v = 0$ at $u = 3$ and $\left(dv/du\right) = -5/3$ at $(u, v) = (3, 0)$. Explain why a second-order differential equation could be reduced to a first-order equation in terms of u and v. Integrate equation (2.47) numerically and plot v as a function of u.

(c) It is clear from the analysis in part (a) that an isothermal sphere extends to infinity and has an infinite amount of mass. In order to obtain a more realistic solution, we may assume that the system is bounded by a spherical box of radius R. We can now characterize the isothermal sphere by a dimensionless parameter $\lambda \equiv (RE/GM^2)$. Show that an isothermal sphere cannot exist if $\lambda < \lambda_{crit}$, where $\lambda_{crit} \cong -0.335$. In other words, the entropy of a self-gravitating system has an extremum only if $\lambda > \lambda_{crit}$.

2.11 Gravity and degeneracy pressure [2]

The description in the last few problems relates to systems of point particles. Another class of systems which are of interest in astrophysics are self-gravitating 'fluid' systems described by an equation of state of the form $P = P(\rho)$. A degenerate Fermi gas of electrons or neutrons, for example, can be described by such an equation of state. If the degeneracy pressure can balance the inward pull of gravity, then it is possible to obtain a stable static configuration. Such configurations are of relevance in the study of the final stages of stellar evolution. The purpose of this exercise is to derive the behaviour of systems supported by the degeneracy pressure.

(a) As the material of the star is compressed, the individual atoms lose their identity and – if the temperature is not too high – the electrons become a degenerate Fermi gas. The equation of state of an ideal, degenerate Fermi gas was obtained in problem 1.12. It can be written as

$$P_{nr} = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m}\left(\frac{Z}{A}\right)^{5/3}\left(\frac{\rho}{m_p}\right)^{5/3} \equiv \lambda_{nr}\rho^{5/3} \tag{2.48}$$

in the non-relativistic limit, and

$$P_r = \frac{1}{4}(3\pi^2)^{1/3}\hbar c\left(\frac{Z}{A}\right)^{4/3}\left(\frac{\rho}{m_p}\right)^{4/3} \equiv \lambda_r\rho^{4/3} \tag{2.49}$$

in the relativistic limit. Here m is the mass of the electron and m_p is the mass of the proton. Show that such a gas becomes *more and more* ideal as the density *increases* and satisfies the condition:

$$\rho \gg \left(\frac{me^2}{\hbar^2}\right)^3\left(\frac{m_pA}{Z}\right)Z^2. \tag{2.50}$$

(b) Further, show that, for a spherically symmetric system in which pressure balances gravity, the density ρ satisfies the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{1/3}}\frac{d\rho}{dr}\right) = -\frac{12\pi G}{5\lambda_{nr}}\rho \tag{2.51}$$

in the non-relativistic case, and

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{2/3}}\frac{d\rho}{dr}\right) = -\frac{3\pi G}{\lambda_r}\rho \tag{2.52}$$

in the relativistic case.

(c) Consider the solution to the non-relativistic case first. Argue from dimensional considerations that the solution must have the form $\rho(r) \propto R^{-6} f(r/R)$, where R is the radius of the star. Hence, show that for such systems $R \propto M^{-1/3}$ and $\bar{\rho} \propto M^2$. With a suitable transformation of variables, reduce the equation and the boundary conditions to the form

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -y^{3/2}, \quad y'(0) = 0, \quad y(1) = 0. \tag{2.53}$$

Numerical integration shows that $y(0) \cong 178$ and $y'(1) \cong -132$. From these values show that

$$MR^3 \cong \frac{92\hbar^6}{G^3 m^3 m_{\rm p}^5}\left(\frac{Z}{A}\right)^5. \tag{2.54}$$

(d) In the relativistic case, argue that the solution must have the form $\rho(r) \propto R^{-3}$, $f(r/R)$. Hence, show that $M = M_0$, some fixed constant. Convert the equation into the form

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -y^3, \quad y'(0) = 0, \quad y(1) = 0. \tag{2.55}$$

Numerical integration gives $y(0) \cong 6.9$ and $y'(1) \cong -2.0$. Using these values show that

$$M_0 \cong \frac{3.1}{m_{\rm p}^2}\left(\frac{Z}{A}\right)^2\left(\frac{\hbar c}{G}\right)^{3/2} \cong 5.8\left(\frac{Z}{A}\right)^2 \mathrm{M}_\odot \tag{2.56}$$

if the gas is made up of neutrons. (For $A \cong 2Z$, $M_0 \cong 1.45\,\mathrm{M}_\odot$.)

2.12 Collisional evolution of gravitating systems [3]

The gravitational force acting on any particle in a self-gravitating system can be divided into two parts, $\mathbf{f}_{\rm sm} + \mathbf{f}_{\rm fluc}$. The $\mathbf{f}_{\rm sm}$ is due to the gravitational potential arising from the smooth distribution of matter. Since the matter is made up of individual particles, there will be a deviation from the smooth force $\mathbf{f}_{\rm sm}$, and this deviation is denoted by the fluctuating part of the force $\mathbf{f}_{\rm fluc}$. The latter part produces a slow diffusion of particles in the momentum space. This process is called 'soft collisions' and the time-scale for this process was estimated in problem 1.14.

In this exercise, we shall derive the equation satisfied by the distribution function $f(\mathbf{x}, \mathbf{p}, t)$ describing the system, taking into account the slow diffusion in the momentum space due to soft collisions. In general, such a diffusion process can be studied by an equation of the following kind:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} - \nabla\phi\cdot\frac{\partial f}{\partial \mathbf{v}} = -\frac{\partial J^\alpha}{\partial p^\alpha}. \tag{2.57}$$

The right-hand side is the divergence of a particle current J^α in the momentum space which is characteristic of diffusive process. We shall now determine the form of J^α.