Problem Set 8: Big Bang, Structure Formation, and Spherical Collapse

Astrophysics and Cosmology through Problems

Due 20 Nov. 2008

1. Statistical mechanics in an expanding universe, Padmanabhan 6.6

2. Consider a spherically symmetric slightly overdense region within a spatially flat ($k = 0$) homogenous matter-dominated universe.

   (a) Assuming that the region obeys the Friedmann equations with a small positive curvature ($0 < k \ll (aH)^2$), find out how overdensity:

   $$\delta(t) \equiv \frac{\rho(t) - \rho_b(t)}{\rho_b(t)} \tag{1}$$

   evolves with time, where $\rho_b(t)$ is the background density and ρ is the density of the overdense region.

   (b) Can you combine the Newtonian Poisson equation, with the linearized continuity and Newton's 2nd law in an expanding Universe to obtain the same result?

   (c) Now assume that the cosmic density is dominated by vacuum energy: $\rho_{vac} = -p_{vac} =$ const., at late times. Numerically solve, and plot the evolution of $\delta(t)$ and $\delta(a)$ through the transition from matter-dominated to the vacuum-dominated era ($a(t)$ is the cosmic scale factor).

3. Spherically symmetric evolution, Padmanabhan 7.10

As an example of the use of this formula, consider the absorption of radiation by neutral hydrogen at a wavelength of 21 cm. The absorption cross-section for this process is

$$\sigma(\nu) = \frac{A}{4\pi}\left(\frac{3}{4}\right)\left(\frac{h\nu}{2kT_{\rm sp}}\right)\left(\frac{c}{\nu}\right)^2 \delta(\nu - \nu_{\rm H}), \tag{6.22}$$

where $T_{\rm sp}$ is the so-called 'spin-temperature.' It is defined by the relation

$$\frac{n_{\rm up}}{n_{\rm down}} = \exp\left(-\frac{h\nu_{\rm H}}{kT_{\rm sp}}\right), \tag{6.23}$$

where $n_{\rm up}$ and $n_{\rm down}$ denote the number of atoms in the upper and lower energy levels. Estimate the optical depth due to 21 cm absorption.

6.6 Statistical mechanics in an expanding universe [3]

The discussion in problem 6.3 shows that the universe was radiation dominated at redshifts higher than $z_{\rm eq} \cong 3.9 \times 10^4 \Omega h^2$. In the radiation dominated phase, the temperature will be greater than $T_{\rm eq} \cong 9.2\Omega h^2$ eV and will be increasing as $(1+z)$. As we go to earlier phases of the universe the radiation will produce particle – antiparticle pairs of different kinds. These elementary particles will be interacting with each other via different processes which will try to maintain statistical equilibrium between the particles. In order to study the early phases of the universe, one would like to understand how the material content of the universe changes as it expands.

(a) Consider some early epoch at which the universe was populated by different species of particles such as electrons, positrons, muons, neutrinos, etc. Let the distribution function for the particle species A be $f_A(\mathbf{x}, \mathbf{p}, t)$. Determine the conditions under which we may assume these particles to be in statistical equilibrium with (i) other particles and (ii) radiation.

If the species A is in statistical equilibrium, then we can take the distribution function to be

$$f_A(\mathbf{p}, t)\, d^3\mathbf{p} = \frac{g_A}{(2\pi)^3}\left\{\exp\left[\left(E_{\rm p} - \mu_A\right)/T_A(t)\right] \pm 1\right\}^{-1} d^3\mathbf{p}, \tag{6.24}$$

where g_A is the spin degeneracy factor of the species, $\mu_A(T)$ is the chemical potential, $E(\mathbf{p}) = \left(\mathbf{p}^2 + m^2\right)^{1/2}$ and $T_A(t)$ is the temperature characterizing this species at time t. The upper sign $(+1)$ corresponds to fermions and the lower sign (-1) is for bosons. Should the temperature be the same for all species and photons, that is, should $T_A = T_B = \ldots = T$?

(b) Express the number density of particles, (n), energy density (ρ) and pressure (p) as integrals over the distribution function $f(\mathbf{p})$. Using these expressions show that

$$d\left(sa^3\right) \equiv d\left\{\frac{a^3}{T}(\rho + P - n\mu)\right\} = \left(\frac{\mu}{T}\right) d\left(na^3\right). \tag{6.25}$$

Argue that the quantity s, defined by the first equation can be interpreted as the entropy density of the universe. Under what conditions will the expansion of the universe be treated as adiabatic?

(c) Show that the energy density, number density and entropy density contributed by highly relativistic ($T \gg m$) and non-degenerate ($T \gg \mu$) particles can be written as

$$\rho_{\rm rad} = g_{\rm rad}\left(\frac{\pi^2}{30}\right)T^4, \quad n = \lambda\left(\frac{\zeta(3)}{\pi^2}\right)T^3, \quad s = q\left(\frac{2\pi^2}{45}\right)T^3, \tag{6.26}$$

where $\zeta(m)$ is a Riemann zeta function of order m and

$$g_{\rm rad} = \sum_{\rm B} g_{\rm B}\left(\frac{T_{\rm B}}{T}\right)^4 + \sum_{\rm F}\frac{7}{8}g_{\rm F}\left(\frac{T_{\rm F}}{T}\right)^4. \tag{6.27}$$

$$\lambda = \sum_{\rm B} g_{\rm B}\left(\frac{T_{\rm B}}{T}\right)^3 + \frac{3}{4}\sum_{\rm F} g_{\rm F}\left(\frac{T_{\rm F}}{T}\right)^3. \tag{6.28}$$

$$q = \sum_{\rm B} g_{\rm B}\left(\frac{T_{\rm B}}{T}\right)^3 + \frac{7}{8}\sum_{\rm F} g_{\rm F}\left(\frac{T_{\rm F}}{T}\right)^3. \tag{6.29}$$

Here $T_{\rm B}, T_{\rm F}$ refer to the temperatures characterizing the distribution functions of specific species of boson or fermion. What is the number density when $T \ll m$?

Also show that, during the radiation dominated era, the temperature of the universe at time t is given by the equation

$$t \cong 0.3 g^{-1/2}\left(\frac{m_{\rm pl}}{T^2}\right) \cong 1{\rm s}\left(\frac{T}{1\,{\rm MeV}}\right)^{-2} g^{-1/2}. \tag{6.30}$$

(d) Consider a species of particles which was in equilibrium with the rest of the matter in the universe up to a time $t = t_{\rm D}$ and 'decouples' at $t = t_{\rm D}$. For $t > t_{\rm D}$ we may assume that each of the particles of this species moves along a geodesic. Determine the distribution function for such a species at $t \gg t_{\rm D}$. In particular, discuss the form of the distribution function if (i) the particles decouple when they are extremely relativistic ($T_{\rm D} \gg m$); or (ii) the particles decouple when they are non-relativistic ($T_{\rm D} \ll m$). What is the energy density contributed by the decoupled species in either of the above cases?

6.7 Relics of relativistic particles [2]

Consider the composition of the universe at a temperature slightly lower than 10^{12} K. We will assume that the universe is populated by electrons (e), positrons ($\bar{\rm e}$), three species of massless neutrino (ν_e, ν_μ, ν_τ), neutrons (n), protons (p) and photons (γ).

(a) Estimate the number density of p, n, e, $\bar{\rm e}$, ν_e, ν_μ and ν_e relative to the photons γ at this temperature.

7.10 Spherically symmetric evolution [2]

In the non-linear regime when $\delta \gtrsim 1$ it is not possible to solve equations for pressureless dust exactly. Some progress, however, can be made if we assume that the solutions are spherically symmetric. We consider, at some initial epoch t_i, a spherical region of the universe which has a slight constant overdensity compared to the background. As the universe expands, the overdense region will expand more slowly compared to the background, will reach a maximum radius, contract and virialize to form a bound non-linear system. Such a model is called 'spherical top-hat'.

(a) Show that in the case of spherical symmetry the density contrast δ satisfies the equations

$$\delta'' + \frac{3A}{2b}\delta' = \frac{4}{3}\frac{(\delta')^2}{(1+\delta)} + \frac{3A}{2b^2}\delta(1+\delta), \quad A = \left(\frac{\rho_b}{\rho_c}\right)\left(\frac{\dot{a}b}{a\dot{b}}\right)^2, \tag{7.49}$$

or

$$\ddot{\delta} - \frac{4}{3}\frac{\dot{\delta}^2}{(1+\delta)} + \frac{2\dot{a}}{a}\dot{\delta} = 4\pi G\rho_b\delta\,(1+\delta)\,, \tag{7.50}$$

where the dot denotes (d/dt) and the prime denotes (d/db); here $b(t)$ is the growing solution to the linear perturbation equation. Further, show that the solution to (7.50) can be expressed in the form

$$\delta(t) = \frac{2GM}{\Omega_m H_0^2 a_0^3}\left[\frac{a(t)}{R(t)}\right]^3 - 1 \equiv \lambda\left(\frac{a}{R}\right)^3 - 1\,, \tag{7.51}$$

where M is a constant and $R(t)$ satisfies the equation

$$\ddot{R} = -\frac{GM}{R^2} - \frac{4\pi G}{3}(\rho + 3p)_{\rm rest}\,R\,, \tag{7.52}$$

in which $(\rho + 3p)_{\rm rest}$ is due to components other than dust-like matter. Interpret this equation.

(b) Assume that the background is described by a $\Omega = 1$, matter dominated universe for which $b = a = (t/t_0)^{2/3}$. In this case, show that the evolution of density contrast can be expressed in the form $1 + \delta(a) = \left(2GM/H_0^2\right)\left[a^3/R^3(a)\right]$, where $R(t)$ satisfies the equation

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2} \tag{7.53}$$

and M is a constant. Let $R(t_i) = r_i$ where t_i is the time at which initial conditions are specified. Let $M = (4\pi/3)\,r_i^3\rho_b(t_i)(1+\delta_i)$, with $\delta_i > 0$. Further assume that $\left(\dot{R}^2/2\right)_{t=t_i} = (1/2)\,H_i^2r_i^2$.

(i) Show that the evolution of the density contrast $\delta(z)$ is given implicitly by the functions $[\delta(\theta), z(\theta)]$ with

$$(1+z) = \left(\frac{4}{3}\right)^{2/3} \frac{\delta_i(1+z_i)}{(\theta - \sin\theta)^{2/3}} = \left(\frac{5}{3}\right)\left(\frac{4}{3}\right)^{2/3} \frac{\delta_0}{(\theta - \sin\theta)^{2/3}}, \qquad (7.54)$$

$$\delta = \frac{9}{2}\frac{(\theta - \sin\theta)^2}{(1-\cos\theta)^3} - 1 \qquad (7.55)$$

where δ_i is the density contrast at the initial redshift z_i and $\delta_0 = (3/5)\delta_i(1+z_i)$ is the corresponding density contrast at the present epoch if the density perturbation grows according to linear theory. Also show that the density contrast at any epoch, predicted by the linear theory, is

$$\delta_L = \frac{3}{5}\left(\frac{3}{4}\right)^{2/3}(\theta - \sin\theta)^{2/3}. \qquad (7.56)$$

(ii) Compare the results for δ and δ_L at different epochs to estimate the accuracy of linear theory.

(iii) Prove that the spherical region reaches a maximum radius of $r_m = (3x/5\delta_0)$ at a redshift z_m such that $(1+z_m) \cong (\delta_0/1.062)$ and the density contrast at maximum expansion is about 4.6.

(c) After reaching the maximum expansion, the matter will collapse inwards. It is likely that the system will become virialized during the collapse and form a gravitationally bound object. (i) Estimate the redshift z_{coll} at which the virialization occurs and (ii) the density of the collapsed object.

7.11 Scaling laws for spherical evolution [3]

(a) Explain qualitatively how the evolution described in parts (c) and (d) of the last problem becomes modified if the initial density is not constant in the spherical region but is given by some profile $\rho_i(r)$ at $t = t_i$.

(b) Assume that the initial density profile is such that the energy $E(M)$ of a shell containing mass M is given by a power law $E(M) = E_0(M/M_0)^{2/3-\epsilon}$. Describe the final density profile assuming that the evolution is self-similar.

7.12 Self-similar evolution of Vlasov equation [2]

At the next level of approximation, one would like to obtain non-linear solutions to the collisionless Boltzmann equation. Needless to say, this is far more difficult than in the case of pressureless dust. Some amount of progress can be made by assuming that the evolution is self-similar in a finite region.

(a) Show that the Vlasov equation (see problem 7.3) admits self-similar solutions of the form

$$f = t^{-(3\alpha+1)}\hat{f}\left(\frac{\mathbf{x}}{t^{\alpha}}, \frac{\mathbf{p}}{t^{\alpha+1/3}}\right). \qquad (7.57)$$