Problem Set 7: Friedmann Cosmology

Astrophysics and Cosmology through Problems

Due 13 Nov. 2008

1. The dynamics of gravitational metric is described by the Einstein equation:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1}$$

where $G_{\mu\nu}$ is known as the Einstein tensor, $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ is the Ricci tensor and $R = R^{\mu}_{\mu}$ is known as the Ricci scalar. All these tensors can be computed for arbitrary metrics, using standard packages in *Maple* or *Mathematica*. $T_{\mu\nu}$ is the energy-momentum tensor of the matter.

   (a) Show that this equation leads to Poisson equation for weak (Newtonian) gravity (Eq. 5.8 in Padmanabhan).

   (b) Show that Einstein tensor vanishes for Schwarzschild metric (Eq. 5.41 in Padmanabhan).

   (c) The most general spatially homogeneous and isotropic metric, known as the Friedmann-Robertson-Walker (FRW) metric, is:

$$ds^2 = dt^2 - a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta \; d\phi^2) \right]. \tag{2}$$

   Calculate the components of the Einstein equation for this metric, in a universe filled with spatially homogeneous density and pressure $\rho(t)$ and $p(t)$. These are known as the Friedmann equations.

   (d) Combine the Friedmann equations to derive the first law of thermodynamics for an adiabatic system:

$$d[\rho(t) a^3(t)] + p(t) d[a^3(t)] = 0. \tag{3}$$

   This is known as the continuity equation.

2. Find $a(t)$ in a universe filled with

   (a) dust $(p, k = 0)$

   (b) radiation $(p = \rho/3, k = 0)$

   (c) curvature $(\rho = p = 0, k < 0)$

3. Show that 3-momentum of free particles (moving on geodesics) decays as a^{-1} in an expanding FRW metric. How do the temperatures of relativistic and non-relativistic gases depend on a?