Problem Set 6: Introduction to General Relativity

Astrophysics and Cosmology through Problems

Due 30 Oct. 2008

1. Accelerated frames, special relativity and gravity, Padmanabhan 5.1: parts (b) and (d)
2. Gravity and the metric tensor, Padmanabhan 5.2
3. Particle trajectories in a gravitational field, Padmanabhan 5.3: parts (b), (c), and (d)
4. Schwarzschild metric, Padmanabhan 5.11: parts (b) and (c)

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General relativity

General relativity is developed through problems in this chapter. All the key ideas are introduced *ab initio* and no previous exposure is assumed. It requires, however, a fair amount of practice to become fully conversant with the manipulation of tensor indices. Exposure to a more conventional course in general relativity will help in this matter.

We use a line element with the signature $(+ - - -)$. The Latin indices $(i, k, ...,$ etc.) go over 0,1,2,3, while the Greek indices $(\alpha, \beta, ...,$ etc.) go over 1,2,3. Summation over repeated indices is assumed.

5.1 Accelerated frames, special relativity and gravity [2]

Combining special relativity with Newtonian gravity turns out to be far more difficult than one would have imagined *a priori*. Any attempt to do so inevitably suggests a geometrical description for gravity. In this problem we shall examine some simple attempts to combine Newtonian gravity and special relativity and see how a geometrical description arises in the process.

(a) Let (T, X, Y, Z) be an inertial coordinate system. Consider an observer moving along the X-axis of this frame with a constant acceleration g. We will construct a coordinate system (t, x, y, z) for the accelerated observer by the following procedure. (i) The observer will use the proper time τ shown by a clock carried by him (her) for his (her) measurements. (ii) Let $\mathcal{P}$ be some event in the spacetime to which the observer has to attribute coordinates (t, x, y, z). Since the motion is along the X-axis the observer can set $y = Y$, $z = Z$. To attribute the (t, x) coordinates he (she) proceeds as follows. He (she) sends a light signal at time $\tau = t_A$ to this event. The light signal is reflected at the event $\mathcal{P}$ and the observer receives it back at time $\tau = t_B$. The observer then attributes to the event $\mathcal{P}$ the time coordinate $t = (t_A + t_B)/2$ and space coordinate $x = (t_B - t_A)/2$.

Use this criterion to determine the coordinate transformation between the accelerated and inertial observers to be

$$X = g^{-1} e^{gx} \cosh gt, \quad T = g^{-1} e^{gx} \sinh gt. \tag{5.1}$$

Show that one can introduce another space coordinate $\bar{x}$ such that the transformation becomes

$$(1 + gX) = (1 + g\bar{x})\cosh gt, \qquad gT = (1 + g\bar{x})\sinh gt, \quad Y = y; \quad Z = z. \tag{5.2}$$

These transformations are clearly non-linear and hence do not preserve the form of the line element ds^2. Show that the line elements in the inertial and accelerated frames are related by

$$\begin{aligned} ds^2 &= dT^2 - dX^2 - dY^2 - dZ^2 = e^{2gx}\left(dt^2 - dx^2\right) - dy^2 - dz^2 \\ &= (1 + g\bar{x})^2\, dt^2 - d\bar{x}^2 - dy^2 - dz^2 . \end{aligned} \tag{5.3}$$

This result shows that non-inertial frames are described by intervals of the form $ds^2 = g_{ik}(x)\, dx^i dx^k$, where $g_{ik}(x)$ – called the metric tensor – will, in general, depend on t and $\mathbf{x}$.

(b) The action for a particle of mass m, in an external gravitational field characterized by the Newtonian potential $\phi(t, \mathbf{x})$, can be taken to be

$$A = \int dt\, \frac{1}{2} m v^2 - \int dt\, m\phi \tag{5.4}$$

in Newtonian gravity. The simplest generalization of this action, in the context of special relativity, will be

$$A = -mc^2 \int dt \sqrt{1 - \frac{v^2}{c^2}} - \int dt\, m\phi . \tag{5.5}$$

Argue that an action of this form leads to the following conclusions:

(i) The gravitational field is locally indistinguishable from a suitably chosen non-inertial frame of reference.

(ii) The gravitational field affects the rate of flow of clocks in such a way that the clocks slow down in strong gravitational fields. To the lowest order in (ϕ/c^2),

$$\Delta t' = \Delta t\left(1 - \frac{\phi}{c^2}\right), \tag{5.6}$$

where Δt is the time interval measured by a clock in the absence of the gravitational field and $\Delta t'$ is the corresponding interval measured by a clock, located in the gravitational potential.

(c) We know that annihilation of electrons and positrons can lead to γ-rays and that under suitable conditions one can produce e^+e^--pairs from the radiation. Devise a suitable thought experiment based on the above facts to conclude that gravitational field must necessarily affect the rate of flow of clocks. Hence, argue that global inertial frames cannot exist in the presence of gravity.

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(d) Show that the dynamics of a particle in an external, weak ($\phi \ll c^2$) gravitational field can be derived from a purely 'geometrical' action of the form

$$A = -mc \int ds = -mc \int \sqrt{g_{ik}\, dx^i\, dx^k}\,, \tag{5.7}$$

where

$$g_{ik}\, dx^i\, dx^k \cong \left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 - d\mathbf{x}^2\,. \tag{5.8}$$

This result suggests that the Newtonian gravitational field can be thought of as modifying the line interval ds^2. Compare this result with the one obtained in (a) above.

The above results imply that a weak gravitational field cannot be distinguished from a modified spacetime interval as far as mechanical phenomena are concerned.

5.2 Gravity and the metric tensor [2]

The last problem shows that weak gravitational fields can be provided with a geometrical description as far as mechanical phenomena are concerned. Einstein made a bold generalization of this result by postulating that: *all aspects of gravitational physics allow a geometrical description.* In other words, we generalize the tentative conclusion of the preceding problem to include *arbitrarily strong gravitational fields* and *all* possible physical phenomena. We will now explore some elementary consequences of this postulate.

(a) To begin with, we shall generalize the result obtained in problem 5.1(d) and postulate that the gravitational field will modify the spacetime interval to the form

$$ds^2 = g_{ik}(x)\, dx^i\, dx^k\,, \tag{5.9}$$

where the metric tensor $g_{ik}(x)$ characterizes a specific gravitational field. Show that: (i) it is not possible, in general, to change the coordinates from x^i to some other x'^i such that ds^2 in the above equation reduces to the Lorentz form; (ii) it is, however, possible to choose coordinate systems such that, around any event $\mathscr{P}$, the following conditions are satisfied: (1) $g'_{ik}(\mathscr{P}) = \eta_{ik}$; (2) $(\partial g'_{ik}/\partial x^{k'})_{\mathscr{P}} = 0$. Can one make the second derivatives $(\partial^2 g_{ik}/\partial x^a \partial x^b)$ at $\mathscr{P}$ vanish as well?

(b) Since g_{ik} cannot be reduced to any preassigned form, no coordinate system has a preferred status in the presence of a gravitational field. This implies that all laws of physics must be expressed in terms of physical quantities which are coordinate-independent. The simplest such entity is a tangent vector v^i to any curve $x^i(\lambda)$ defined by $v^i = (dx^i/d\lambda)$. All other four-vectors will be *defined* to be quantities which transform similar to the tangent vector v^i. Determine how a vector v^i transforms when the coordinate system is changed from x^i to x'^i. What is the natural transformation law for such higher order tensors as T^{ik}, S^{ijk}, etc.?

(c) Treating the metric tensor g_{ik} as a matrix we can define its inverse g^{ik} such that $g^{ik}g_{kl} = \delta^i_l$. Derive the transformation law for g^{ik}. Let g denote the determinant of g_{ik}. Show that

$$\frac{\partial g}{\partial x^i} = g g^{ab} \left(\frac{\partial g_{ab}}{\partial x^i} \right) = -g g_{ab} \left(\frac{\partial g^{ab}}{\partial x^i} \right). \tag{5.10}$$

(d) We define the operation of 'raising and lowering an index' of a tensor by using the appropriate form of metric tensor. That is, given T^{ik} we define two new tensors $T^i{}_k$ and T_{ik} by the relations

$$T^i{}_k \equiv g_{ka} T^{ia}, \quad T_{ik} \equiv g_{ia} g_{kb} T^{ab}. \tag{5.11}$$

Show that these are valid tensor operations in the sense that if these equations are true in one frame they will be true in all frames.

5.3 Particle trajectories in a gravitational field [3]

(a) In special relativity, the equation of motion for a free particle can be obtained by varying the action

$$A = -m \int ds = -m \int \left(\eta_{ab} \frac{dX^a}{d\lambda} \frac{dX^b}{d\lambda} \right)^{1/2} d\lambda. \tag{5.12}$$

The variation of A gives $\left(d^2X^a/d\lambda^2\right) = \left(du^a/d\lambda\right) = 0$, where $u^a = \left(dX^a/d\lambda\right)$ is the four-velocity which is a tangent vector to the curve $X^a(\lambda)$. This equation can be written equivalently as

$$\frac{du^a}{d\lambda} = \left(\frac{dX^b}{d\lambda} \right) \left(\frac{\partial u^a}{\partial X^b} \right) = u^b \left(\frac{\partial u^a}{\partial X^b} \right) \equiv u^b \left(u^a{}_{,b} \right) = 0. \tag{5.13}$$

Show that this is *not* a valid tensor equation in the sense that the form of this equation will not remain the same in an arbitrary coordinate system.

(b) To obtain the correct equation of motion, which is generally covariant, we could proceed as follows. In any small region around an event, one can always choose an inertial coordinate system. Hence, in such a small region, the form of the action in equation (5.12) remains valid. In that small region $\eta_{ab} = g_{ab}$ and so we can also write the action in the form

$$A = -m \int ds = -m \int \sqrt{g_{ab}\, dx^a\, dx^b} = -m \int \left(g_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} \right)^{1/2} d\lambda. \tag{5.14}$$

Since *this* action is made up of proper tensorial quantities it will remain valid in any coordinate system. Hence, the equations derived from this action will provide us with the correct generalization of equation (5.13).

Vary this action and show that the equation of motion for a particle in an arbitrary spacetime is given by

$$\frac{du^i}{d\lambda} + \Gamma^i_{kl} u^k u^l = 0, \tag{5.15}$$

where

$$\Gamma^i_{kl} = \frac{1}{2} g^{ia} \left(-\frac{\partial g_{kl}}{\partial x^a} + \frac{\partial g_{ka}}{\partial x^l} + \frac{\partial g_{al}}{\partial x^k} \right). \tag{5.16}$$

Show, further, that this equation can be written in the form $u^i u^k{}_{;i} = 0$, where the 'covariant derivative' of the vector u^i is defined by the relation

$$u^a{}_{;b} \equiv u^a{}_{,b} + \Gamma^a_{bc} u^c. \tag{5.17}$$

By its very construction, the covariant derivative will be a tensor, and defines a natural generalization of the ordinary derivative to curved spacetime. How does Γ^i_{kl} transform under a coordinate transformation?

(c) We shall use the definition of covariant derivative in equation (5.17) for any vector field $v^i(x)$. We can generalize the notion of covariant derivative to other tensorial quantities by assuming that (1) the 'chain rule' of differentiation should remain valid; and (2) the covariant derivative of a scalar should be the same as the ordinary derivative. Show, using these criteria, that (i) the covariant derivative of v_i differs from that of v^i only in a sign:

$$v_{i;k} = v_{i,k} - \Gamma^l_{ik} v_l \, ; \tag{5.18}$$

(ii) for a mixed tensor the covariant derivative will appear with positive sign for upper indices and negative sign for lower indices:

$$T^a_{b;c} = T^a_{b,c} + \Gamma^a_{dc} T^d_b - \Gamma^d_{bc} T^a_d \, . \tag{5.19}$$

Similar definitions can be used for any higher rank tensor.

(d) Show that covariant derivatives do not commute. More specifically, show that

$$A^a{}_{;b;c} - A^a{}_{;c;b} = -R^a_{ibc} A^i, \tag{5.20}$$

where

$$R^a_{ibc} = \frac{\partial \Gamma^a_{ic}}{\partial x^b} - \frac{\partial \Gamma^a_{ib}}{\partial x^c} + \Gamma^a_{kb} \Gamma^k_{ic} - \Gamma^a_{kc} \Gamma^k_{ib} \, . \tag{5.21}$$

Is this quantity a tensor?

5.4 Parallel transport [1]

Consider a curve $x^i(\lambda)$ passing through an event $\mathscr{P}$ in spacetime. Let the coordinates of $\mathscr{P}$ be $x^i(0)$. If $v^j(x)$ is a vector field, then we can interpret the quantity $v^j{}_{;i}\,(dx^i/d\lambda)$ to be the change of the vector field along a direction specified by the tangent vector $(dx^i/d\lambda)$. Show how this concept can be used to generate a vector *field* from a *given* vector at $\mathscr{P}$. (This process is called 'parallel transport'.)

the metric outside this region (i.e. for $r > R$) has the form

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right). \qquad (5.41)$$

Also determine the metric due to a spherical shell of matter both inside and outside the shell.

(b) Consider the motion of a material particle of mass m in this metric. Show that the orbit of the particle can be determined from the equations

$$\left(1 - \frac{2GM}{r}\right)^{-1} \frac{dr}{dt} = \frac{1}{\mathscr{E}} \left[\mathscr{E}^2 - V^2(r)\right]^{1/2}, \quad r^2\dot{\phi} = \left(\frac{L}{\mathscr{E}}\right)\left(1 - \frac{2GM}{r}\right), \qquad (5.42)$$

where $\mathscr{E}$ and L are constants and

$$V_{\text{eff}}^2(r) = m^2 \left[\left(1 - \frac{2GM}{r}\right)\left(1 + \frac{L^2}{m^2 r^2}\right)\right]. \qquad (5.43)$$

Plot the effective potential $V_{\text{eff}}(r)$ for different values of L and describe qualitatively the nature of orbits for various values of $\mathscr{E}$ and L.

(c) Consider an ultrarelativistic particle which is moving past a body of mass M with velocity $v \lesssim c$. Let the impact parameter be R and the angle of deflection be $\delta\phi$. Calculate $\delta\phi$. Show that $\delta\phi$ has twice the value it would have in the case of Newtonian gravity in the ultrarelativistic limit of $v = c$. Also estimate the corresponding deflection if the force of interaction between the two particles is electromagnetic rather than gravitational. What happens to the deflection as $v \to c$ in the case of electromagnetic interaction?

5.12 Gravitational lensing [2]

Light rays reaching us from distant cosmological sources are deflected by the gravitational field of intervening masses. This could lead, under favourable conditions, to the formation of multiple images of the source. Assume that all the deflection takes place when the light crosses the 'deflector plane', which is defined to be the plane containing the deflecting object and is perpendicular to the line connecting the source and the observer. It is convenient to project all relevent quantities on to this two-dimensional plane. Let $\mathbf{s}$ and $\mathbf{i}$ denote the two-dimensional vectors giving the source and image positions on this two-dimensional plane; and let $\mathbf{d}(\mathbf{i})$ be the (vectorial) deflection produced by the lens.

(a) Show that the image and source positions are related by the equation

$$\mathbf{s} = \mathbf{i} - \frac{D_{\text{LS}}}{D_{\text{OS}}} \mathbf{d}(\mathbf{i}), \qquad (5.44)$$

where L, S and O stand for lens, source and observer and D_{LS} is the distance between the lens and the source, etc.