3.10 Shock waves [2]

In the last two problems we considered situations in which gas flow can result in nearly discontinuous changes in the parameters. This occurs generically when the flow velocity changes from supersonic to subsonic values in some regions. Such a nearly discontinuous transition in density, pressure, etc., is called a shock.

Shocks arise in several astrophysical contexts in which there is a supersonic flow of gas relative to a solid obstacle. (In the laboratory frame, this could arise either because of an object moving through a fluid with supersonic velocities or because gas flowing with supersonic velocities suddenly encounters an obstacle.) Because of the boundary conditions near the solid body, the flow has to change from supersonic to subsonic somewhere near the object. Further, this information cannot reach upstream where the flow is supersonic. Hence, the pattern of flow changes very rapidly over a short region, resulting in a shock wave. In the frame in which the gas is at rest, we may consider the solid body to be moving with supersonic speed and pushing the gas. This necessarily leads to steepening of density profile at some place ahead of the body. We shall consider some of the properties of shock waves in this problem.

(a) Idealize the surface of discontinuity as a plane with gas flow occurring perpendicular to it. Let p_1, ρ_1, v_1 denote the state of the gas to the left of the shock front and let p_2, ρ_2, v_2 give the corresponding values on the right side. Assume that the orientation of gas flow is such that the flow is supersonic on the left side. Using the equations of flow for an ideal polytropic fluid with index γ, relate the ratios $(p_2/p_1), (\rho_2/\rho_1), (v_2/v_1)$ and (T_2/T_1) to the index γ and the Mach number $M_1 = (v_1/c_s)$ on one side.

(b) Show that, under the conditions of the above problem, $p_2 > p_1$, $\rho_2 > \rho_1$, $T_2 > T_1$, $v_2 < v_1$ and that the maximum possible increase in the density is by a factor $(\gamma + 1)/(\gamma - 1)$.

(c) In the above analysis, it was assumed that the discontinuity occurs at a very short region, so that it can be treated as infinitesimal for mathematical purposes. Estimate the order of magnitude of various terms in the equations of fluid motion with dissipation, and show that the discontinuity occurs over a region of the order of mean-free-path.

(d) A gas in a semi-infinite cylindrical pipe is terminated by a piston at one end. At $t = 0$, the piston begins to move towards the positive x-axis with a constant speed U. Determine the resulting gas flow.

3.11 Particle acceleration mechanisms [2]

Observations suggest that the energy spectrum of particles in many astrophysical contexts has a power law form. To generate and maintain such a power law, one often requires mechanisms which will accelerate the particles. We shall explore some general features of such acceleration mechanisms in this problem.

by a photon) is

$$\frac{dE}{dt} = \frac{4}{3}\sigma_T c U_{\rm rad}\gamma^2 \left(\frac{v}{c}\right)^2 \equiv P_{\rm inComp}\,, \tag{4.35}$$

where $U_{\rm rad} = aT^4$ is the energy density of the radiation field. In the non-relativistic limit, for electrons with temperature T_e, this gives

$$\left(\frac{dE}{dt}\right)_{\rm nr} \cong \frac{4}{3}\sigma_T c U_{\rm rad}\left(\frac{3kT_e}{m_e c^2}\right). \tag{4.36}$$

Show that the average fractional energy gained by a photon per collision is

$$\left\langle\frac{\Delta\epsilon}{\epsilon}\right\rangle = \frac{\langle\Delta\epsilon\rangle}{\hbar\omega_i} = \frac{4}{3}\gamma^2\left(\frac{v}{c}\right)^2 = \begin{cases} (4/3)\gamma^2 & (\text{if} \quad v \cong c) \\ (4kT_e/m_e c^2) & (\text{if} \quad v \ll c) \end{cases}. \tag{4.37}$$

Combining with (4.34) we find that the net energy change of photons in Compton scattering with a thermal distribution of electrons with temperature T_e is

$$\left\langle\frac{\Delta\epsilon}{\epsilon}\right\rangle = -\frac{\hbar\omega_i}{m_e c^2} + \frac{4kT_e}{m_e c^2}, \tag{4.38}$$

for $v \ll c$.

The process described above acts as a major source of cooling for relativistic plasma as well as a mechanism for producing high energy photons. The time-scale for 'Compton cooling' of individual relativistic particle is

$$t_{\rm Comp\,cool} \cong \frac{\gamma mc^2}{P_{\rm inComp}} \cong 4\times10^{-3}\gamma^{-1}\beta^{-2}\left(\frac{T_{\rm rad}}{10^6{\rm K}}\right)^{-4}\ {\rm s}, \tag{4.39}$$

where $T_{\rm rad}$ is the radiation temperature. In the case where electrons are non-relativistic with temperature T_e, this time-scale is

$$t_{\rm Comp\,cool} \cong \frac{kT_e}{P_{\rm inComp}} \cong \frac{1}{n_\gamma\sigma_T}\left(\frac{m}{T_{\rm rad}}\right) = 1.3\times10^{-3}\left(\frac{T_{\rm rad}}{10^6\,{\rm K}}\right)^{-4}\ {\rm s}. \tag{4.40}$$

4.10 Comptonization [3]

The energy transfer between electrons and photons takes place through Compton scattering. This process dominates when (i) scattering is more important than true absorption; and (ii) the temperature of the low density electron gas is far higher than the Planck distribution with the same energy. In that case, there is net energy transfer from the electrons to the photons which distorts the original photon spectrum. We will now work out the details of this process.

(a) In a scattering of a photon of frequency ν with an electron of energy E, resulting in a photon of frequency ν' and an electron of energy E', the energy transfer is

$$h\left(\nu' - \nu\right) \equiv h\Delta \cong -\left(\frac{h\nu}{mc}\right)\mathbf{p}\cdot\left(\hat{\mathbf{n}} - \hat{\mathbf{n}}'\right), \tag{4.41}$$

where $\mathbf{p}$ is the initial momentum of the electron and $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$ are the directions of the initial and final photons. This is the lowest order result for $(\Delta\nu/\nu)$ and is $\mathcal{O}(v/c)$. Prove this result.

(b) The evolution equation for photon number density in a homogeneous medium is

$$\frac{\partial n(\nu)}{\partial t} = \int d^3p \int d\Omega \left(\frac{d\sigma}{d\Omega}\right) c \left[n(\nu)\left(1+n(\nu')\right)N(E) - n(\nu')\left(1+n(\nu)\right)N(E')\right], \tag{4.42}$$

where $(d\sigma/d\Omega)$ is the electron–photon scattering cross-section, $n(\nu)$ is the photon distribution function and $N(E)$ is the electron distribution function. Explain the origin of each of the terms in this equation.

(c) The integro-differential equation in (4.42) can be approximated as a differential equation when $\Delta \ll 1$. (This is similar to the approach used in problem 2.12.) Expand $n(\nu') = n(\nu + \Delta)$ and $N(E') = N(E - h\Delta)$ in a Taylor series in Δ, retaining up to quadratic order. Assuming that $N(E)$ is a Maxwellian distribution with temperature T, show that equation (4.42) can now be written as

$$\begin{aligned}\frac{\partial n}{\partial t} &= \left(\frac{h}{kT}\right)\left(\frac{\partial n}{\partial x} + n(n+1)\right) I_1 \\ &+ \frac{1}{2}\left(\frac{h}{kT}\right)^2 \left(\frac{\partial^2 n}{\partial x^2} + 2(1+n)\frac{\partial n}{\partial x} + n(n+1)\right) I_2,\end{aligned} \tag{4.43}$$

where $x = (h\nu/kT)$ and I_1 and I_2 are the integrals

$$I_1 = \int d^3p\, d\Omega \left(\frac{d\sigma}{d\Omega}\right) cN(E)\,\Delta, \tag{4.44}$$

$$I_2 = \int d^3p\, d\Omega \left(\frac{d\sigma}{d\Omega}\right) cN(E)\,\Delta^2. \tag{4.45}$$

(d) Use the expression for average energy transfer $\Delta E = (h\nu/mc^2)(h\nu - 4kT)$ (see equation (4.38)) to show that

$$I_1 = (\sigma_T n_e)\left(\frac{h\nu}{mc^2}\right)(4-x) = (\sigma_T n_e)\left(\frac{kT}{mc^2}\right) x(4-x). \tag{4.46}$$

(e) Using (4.41) evaluate I_2 and show that

$$I_2 = 2\left(\frac{\nu}{mc}\right)^2 (kT)(mc)\, n_e \sigma_T. \tag{4.47}$$

Hence, arrive at the final form of the equation (called Kompaneet's equation):

$$\frac{\partial n}{\partial y} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left(\frac{\partial n}{\partial x} + n^2 + n\right)\right], \tag{4.48}$$

where

$$y = t\left(\frac{kT}{mc^2}\right) n_e \sigma_T c. \tag{4.49}$$

(f) In most of the astrophysically interesting cases one can ignore the n^2 term with respect to n. The evolution of the total energy of the photons,

$$E_{\text{pho}}(y) = \frac{2(kT)^4}{h^3c^3} \int_0^\infty n(x)\, x 4\pi x^2 dx, \tag{4.50}$$

can be then determined using (4.48). Show that, if $n(x)$ vanishes suficiently fast for large x, then

$$\frac{dE_{\text{pho}}}{dy} = \frac{8\pi (kT)^4}{(hc)^3} \left[4 \int_0^\infty n x^3 dx - \int_o^\infty n x^4 dx \right]. \tag{4.51}$$

If one can further assume that most of the photon energy is concentrated at $x \ll 1$, then we can ignore the second term with respect to the first on the right-hand side. Show that, in that case, the photon energy, E_{pho}, increases as

$$E_{\text{pho}}(y) \cong E_{\text{pho}}(0) \exp(4y). \tag{4.52}$$

The characteristic e-folding time is

$$t_{\text{Comp}} = \left(\frac{mc^2}{4kT} \right) \left(\frac{1}{n_e \sigma_T c} \right). \tag{4.53}$$

(g) Solve (4.48) when both n and n^2 are ignorable compared to $(\partial n/\partial x)$. Show that the solution is

$$n(x, y) = \frac{1}{(4\pi y)^{1/2}} \int_0^\infty \frac{d\mu}{\mu} n(\mu, 0) \exp\left[-\frac{1}{4y} \left(3y + \ln \frac{x}{\mu} \right)^2 \right]. \tag{4.54}$$

Consider an initial distribution of photons with $n(x, 0) \cong x^{-1}$, which corresponds to the Rayleigh–Jeans limit of the Planck spectrum. Show that this distribution will evolve to

$$n(x, y) \cong x^{-1} e^{-2y} \tag{4.55}$$

due to Comptonization. Hence, the temperature of the radiation will appear to have been diminished by a factor e^{-2y} at the Rayleigh–Jeans end. Such an effect has been observed when microwave background radiation passes through hot intergalactic gas in clusters.

(h) Equation (4.51) can also be used to study the change in the electron energy. Since $(E_{\text{pho}} + E_e)$ should be a constant $(dE_e/dy) = -(dE_{\text{pho}}/dy)$. Calculate (dE_e/dy) when $n(x) = n_0 \exp(-x/\alpha)$. What is the cooling rate of electrons for $\alpha \ll 1$?

4.11 Quantum theory of radiation [3]

In the quantum mechanical treatment, an electromagentic field is treated as a bunch of photons. The emission and absorption of radiation are to be represented as processes which cause transitions between states with different numbers of