Problem Set 4: Instability, Turbulence, and Accretion

Astrophysics and Cosmology through Problems

Due 2 Oct. 2008

1. First, find an approximate formula for viscosity of a gas η as a function of molecular cross-section and other properties of the gas. Then, recalling the definition of Reynolds number (Eq. 3.36 in Padmanabhan), estimate its rough value inside:

   - Amazon river
   - Sun
   - accretion disk around a quasar
   - intergalactic medium of galaxy clusters

2. Instabilities and Turbulence, Padmanabhan 3.7

3. Accretion disk, Padmanabhan 3.13

4. Find the relation between the thickness of an accretion disk and its local temperature

5. As you saw in Problem 1, the molecular viscosity is very small in astrophysical accretion disks, and it is often believed that turbulence is responsible for angular momentum transport. Can you define an effective viscosity for turbulent disks of a given thickness? What qualitative factors could enhance the angular momentum transport?

where $\mathbf{f}_1$ and f_2 are functions which can be determined only by solving the equation of motion. Further, show that the force acting on the solid body moving through the fluid can be expressed as $F = \rho u^2 l^2 f_3(R)$.

3.7 Instabilities and turbulence [3]

The analysis of viscous fluid flow, based on the equations derived above, becomes very complicated in realistic situations. To begin with, exact solutions can be found only with very special assumptions. What is more, these solutions are usually unstable to small perturbations. The growth of small perturbations in a fluid makes the flow complicated and turbulent in realistic situations. In this problem we shall study one of the simplest kinds of instability called Kelvin-Helmholtz instability.

(a) Consider two layers of an incompressible fluid with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. We assume that one layer is attempting to 'slide' on top of another, along the common surface which they share. We shall consider a small region of the fluid on both sides of the surface of separation with fluid velocities tangential to the surface; the surface itself is assumed to be a plane. This kind of a flow, which occurs very often in nature (e.g. wind blowing on the surface of a lake), happens to be unstable to small perturbations. Consider a perturbation in which the surface of discontinuity becomes slightly distorted from the planar shape. Show that, in an ideal incompressible fluid, such an instability will grow. Determine the time-scale for the growth.

(b) Instabilities like the one mentioned above make the fluid flow turbulent in any realistic situation. Explain quantitatively how a complicated turbulent velocity field can arise due to the repeated action of instability like the one described in (a) above.

(c) A highly turbulent flow cannot be analysed quantitatively from the basic equations of fluid mechanics. It is, however, possible to make some progress by introducing reasonable physical assumptions regarding the transport of energy in the turbulent flow. It is usual to assume that fully developed turbulence can be understood in terms of 'eddies' of different sizes coexisting at the same time. Let λ be the size of a generic eddy and let v_λ and ϵ_λ denote the typical change in the velocity (across the eddy) and energy per unit mass contained in the eddy. At very large scales, viscosity does not play any crucial role and the energy which is put into the system is merely transferred to the next level of smaller eddies. We will assume that the rate $\dot{\epsilon}$ at which energy per unit mass is fed from the larger scale to the smaller one is a constant independent of the scale. Let the velocity at the largest scale, L, be U. Show that

$$v_\lambda \cong U \left(\frac{\lambda}{L}\right)^{1/3}. \tag{3.38}$$

This process of energy transfer can go on till the viscous effects become important. Let this happen at some small scale λ_s. Show that $\lambda_s \cong (L/R^{3/4})$, where R is the Reynold's number. Also show that the power spectrum of the velocity field in the range $L^{-1} \ll k \ll \lambda_s^{-1}$ is

$$S(k) \propto \dot{\epsilon}^{1/3} k^{-5/3}. \tag{3.39}$$

This is known as the Kolmogorov spectrum.

3.8 Sound waves [2]

(a) Show that a *compressible* ideal fluid supports the propagation of small disturbances in density and pressure, in the form of a longitudinal wave, with velocity

$$c_s = \left(\frac{\partial p}{\partial \rho}\right)_s^{1/2} = \left(\frac{\gamma p}{\rho}\right). \tag{3.40}$$

What happens to the perturbations in vorticity or entropy?

Also show that: (i) the velocity perturbation is related to the pressure and density perturbation by $v' = (p'/\rho c_s) = (c_s \rho'/\rho)$, where the perturbations are indicated by primed quantities; (ii) the temperature perturbation is related to the velocity perturbation by $T' = c_s \beta T v'/c_p$, where $\beta = (1/V)(\partial V/\partial T)_p$ is the coefficient of thermal expansion.

(b) The dissipative processes will dampen the amplitude of a sound wave propagating through a fluid. Show that, in the lowest order of approximation, the amplitude is attenuated by a factor $\exp(-\gamma x)$, where

$$\gamma = \frac{\omega^2}{\rho c_s^3}\left[\left(\frac{4}{3}\eta + \zeta\right) + \kappa\left(\frac{c_p - c_v}{c_p c_v}\right)\right]. \tag{3.41}$$

(c) If a small disturbance is generated inside the fluid, it will propagate in all directions with the speed of sound, relative to the fluid. When viewed from a fixed coordinate system, the disturbance will move with a velocity which is the sum of sound velocity and the velocity of gas flow $\mathbf{v}$. Different phenomena can arise depending on whether $|\mathbf{v}| > c_s$ or $|\mathbf{v}| < c_s$. Show that (i) if $|\mathbf{v}| < c_s$, then the disturbance can eventually reach all points in the fluid; while (ii) if $|\mathbf{v}| > c_s$, then the disturbance is propagated downstream only inside a conical region with opening angle $\alpha = \sin^{-1}(c_s/|\mathbf{v}|)$.

(d) The analysis in part (a) above assumed that the amplitude of the disturbance was small. But since c_s was proportional to $\rho^{1/3}$, regions of the fluid with higher density will be travelling faster. This will necessarily distort the shape of the wave and cause it to steepen. Such a distortion of shape cannot be understood in linearized theory and we need to study the exact equations.

Consider a one-dimensional fluid flow which is isoentropic. All quantities depend only on x and t and we take $v_x = v(x,t), v_y = v_z = 0$. Show that the

(b) Relate the accretion rate to the density and sound speed of the gas at infinity and show

$$\dot{M} = \pi G^2 M^2 \frac{\rho(\infty)}{c_s^3(\infty)} \left[\frac{2}{5-3\gamma}\right]^{(5-3\gamma)/2(\gamma-1)} . \tag{3.49}$$

Also show how one can determine the velocity and density profiles everywhere.

3.13 Accretion disc [3]

Spherically symmetric accretion can occur only when the angular momentum of the infalling matter is zero. In a more realistic situation one expects the gas to form a disc around the central object and flow in a spiralling pattern. For this process to occur it is necessary for the gas to lose the angular momentum efficiently due to some viscous process.

(a) Consider a viscous accretion disc with an inner radius r_{min} and outer radius r_{max}. It is assumed that: (i) there exists some suitable form of viscous stress which transports angular momentum radially outward; and (ii) the gas flows in nearly Keplerian orbits and spirals inwards slowly. The same viscous stress will also transport energy, some of which will be dissipated locally. Show that, in a steady state, the energy dissipated in an annular ring between r and $(r+dr)$ is

$$| dE | = \frac{3}{2} \dot{M} v_\phi^2(r) \frac{dr}{r} , \tag{3.50}$$

where $\dot{M}$ is the mass flowing through the disc per unit time.

(b) Assume that this energy is radiated from each annular ring in the form of a blackbody with a temperature $T(r)$. The net spectra of the accretion disc will be a superposition of blackbody radiation with different temperatures. Show that the intensity scales approximately as $I_\nu \propto \nu^{1/3}$ in the Rayleigh–Jeans limit.

3.14 The Sedov solution for a strong explosion [3,N]

An explosion can release a large amount of energy E into an ambient gas on a time-scale which is extremely small. This will result in the propagation of a spherical shock wave centred at the location of the explosion. The resulting flow pattern can be determined everywhere if we make the following assumptions:

(i) The explosion is idealized as one which releases energy E *instantaneously* at the origin. The shock wave is taken to be so strong that the pressure p_2 behind the shock is far greater than the pressure p_1 of the undisturbed gas.

(ii) We neglect the original energy of the ambient gas in comparison with the energy E which it acquires as a result of the explosion.

(iii) We take the gas flow to be governed by the equations appropriate for an adiabatic polytropic gas with index γ.

Argue, from dimensional considerations, that the position of the shock front at time t is given by $R(t) \propto \left(Et^2/\rho_1\right)^{1/5}$. Solve the equations of gas flow by