Problem Set 10: Gravitational Radiation and Lensing

Astrophysics and Cosmology through Problems

Due 4 December 2008

I will put some useful notes for gravity waves here, as they are not covered by Padmanabhan. The problems will not require use of all these formulae, but I include them for reference. For weak gravitational fields, the metric of spacetime is nearly that of flatspace; that is,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where $h_{\mu\nu} \ll 1$ and obeys (semicolons are covariant derivatives, commas are partial derivatives):

$$\begin{aligned}
\bar{h}_{\mu\nu} &= h_{\mu\nu} - \frac{1}{2}h^{\alpha}_{\alpha}\eta_{\mu\nu} \\
\Box\bar{h}_{\mu\nu} &= \bar{h}_{\mu\nu\alpha}^{;\alpha} = 0 \\
\bar{h}^{\mu\alpha}_{;\alpha} &= 0 \qquad \text{“Lorentz gauge”}
\end{aligned}$$

$$h_{\mu 0} = 0 \qquad h^{\alpha}_{\alpha} = 0 \quad \text{“Transverse-traceless” gauge}$$

(you use either one gauge or the other, but not both!). The stress-energy tensor for gravity waves is

$$T^{(GW)}_{\mu\nu} = \frac{1}{32\pi}\langle h_{jk,\mu}h^{jk}_{,\nu}\rangle$$

where the brackets denote an averaging over several wavelengths; this formula was derived in TT gauge (see, e.g., Misner-Thorne-Wheeler Sec. 36.7).

The gravity wave power emitted by a nearly-Newtonian, slow-moving ($v \ll c$) source is:

$$L_{GW} = \frac{1}{5}\frac{G}{c^5}\langle\left(\frac{\partial^3}{\partial t^3}I_{jk}\right)\left(\frac{\partial^3}{\partial t^3}I^{jk}\right)\rangle$$

where the brackets here indicate averaging over a few periods of the objects motion (assuming periodicity) and

$$I_{jk} \equiv \int (x_j x_k - \frac{1}{3}\delta_{jk}r^2)\rho d^3x$$

is the *reduced quadrupole moment tensor.*

1. An angry Toronto motorist shakes his fist at another motorist. Using order-of-magnitude estimates, calculate the power of the gravity waves he emits, and calculate what fraction of the total energy he uses goes into gravity waves.

2. Two stars of mass M_1 and M_2 separated by a distance R revolve about each other in a non-relativistic orbit. Due to gravitational radiation dissipating their energy, R evolves in time. Find R(t). How long will it take for their orbit to entirely decay away?

3. Padmanabhan 5.12.

the metric outside this region (i.e. for $r > R$) has the form

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right). \quad (5.41)$$

Also determine the metric due to a spherical shell of matter both inside and outside the shell.

(b) Consider the motion of a material particle of mass m in this metric. Show that the orbit of the particle can be determined from the equations

$$\left(1 - \frac{2GM}{r}\right)^{-1} \frac{dr}{dt} = \frac{1}{\mathscr{E}} \left[\mathscr{E}^2 - V^2(r)\right]^{1/2}, \quad r^2\dot{\phi} = \left(\frac{L}{\mathscr{E}}\right)\left(1 - \frac{2GM}{r}\right), \quad (5.42)$$

where $\mathscr{E}$ and L are constants and

$$V^2_{\rm eff}(r) = m^2 \left[\left(1 - \frac{2GM}{r}\right)\left(1 + \frac{L^2}{m^2 r^2}\right)\right]. \quad (5.43)$$

Plot the effective potential $V_{\rm eff}(r)$ for different values of L and describe qualitatively the nature of orbits for various values of $\mathscr{E}$ and L.

(c) Consider an ultrarelativistic particle which is moving past a body of mass M with velocity $v \lesssim c$. Let the impact parameter be R and the angle of deflection be $\delta\phi$. Calculate $\delta\phi$. Show that $\delta\phi$ has twice the value it would have in the case of Newtonian gravity in the ultrarelativistic limit of $v = c$. Also estimate the corresponding deflection if the force of interaction between the two particles is electromagnetic rather than gravitational. What happens to the deflection as $v \to c$ in the case of electromagnetic interaction?

5.12 Gravitational lensing [2]

Light rays reaching us from distant cosmological sources are deflected by the gravitational field of intervening masses. This could lead, under favourable conditions, to the formation of multiple images of the source. Assume that all the deflection takes place when the light crosses the 'deflector plane', which is defined to be the plane containing the deflecting object and is perpendicular to the line connecting the source and the observer. It is convenient to project all relevent quantities on to this two-dimensional plane. Let $\mathbf{s}$ and $\mathbf{i}$ denote the two-dimensional vectors giving the source and image positions on this two-dimensional plane; and let $\mathbf{d}(\mathbf{i})$ be the (vectorial) deflection produced by the lens.

(a) Show that the image and source positions are related by the equation

$$\mathbf{s} = \mathbf{i} - \frac{D_{\rm LS}}{D_{\rm OS}} \mathbf{d}(\mathbf{i}), \quad (5.44)$$

where L, S and O stand for lens, source and observer and $D_{\rm LS}$ is the distance between the lens and the source, etc.

(b) The deflection produced by the lens can be related to the surface density $\Sigma(\mathbf{x})$ of the intervening mass distribution. Show that

$$\mathbf{d}(\mathbf{i}) = \frac{4GD_{\mathrm{OL}}}{c^2} \int d^2x\, \Sigma(\mathbf{x}) \frac{(\mathbf{i}-\mathbf{x})}{|\mathbf{i}-\mathbf{x}|^2} . \tag{5.45}$$

Describe qualitatively the behaviour of the function $\Sigma(x)$ when the mass distribution is spherically symmetric. Argue that, for spherically symmetric, non-singular mass distributions there will be either one or three images.

(c) Show that the images can be amplified (or de-amplified) with respect to the source. Find an expression for the amplification in terms of the position $\mathbf{s}$ of the source.

(d) Relate the source and image position when the lensing is due to (i) a uniform sheet with surface density Σ_0; (ii) a point mass M; (iii) an isothermal sphere with a small core radius. Calculate the amplification in each of these cases.

(e) Let us choose a coordinate system in the lensing plane such that the source is at the origin. Show that the location of the images can be obtained by calculating the extremum of the function

$$P(\mathbf{i}) = \frac{1}{2} i^2 - \psi(\mathbf{i}) , \tag{5.46}$$

where

$$\psi(\mathbf{i}) = \frac{4GD_{\mathrm{OL}}D_{\mathrm{LS}}}{c^2 D_{\mathrm{OS}}} \int d^2x\, \Sigma(\mathbf{x}) \ln(|\mathbf{i}-\mathbf{x}|) . \tag{5.47}$$